Quantum Field Theory and the Standard Model

Problem solutions

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1 Microscopic theory of radiation

2 Lorentz invariance and second quantization

2.1

2.2

a. In a collision or decay the total energy $E_T = \sum_i E_i$ and total momentum $p_T = \sum_i p_i$ are always conserved. In the center of mass $p_T = 0$ and $E_{cm} = 2E_{proton}$. For a particle with a non-zero mass, the total energy is given by

$$E = \gamma mc^2$$

with $\gamma = \frac{1}{\sqrt{1 - \frac{v_{particle}^2}{c^2}}}$ We therefor have

$$E_{cm} = 2\gamma m_{proton} c^2$$

and thus

$$v = \sqrt{c^2(1 - \frac{1}{\gamma^2})}$$

It gives $v - c = 2.69309841 m.s^{-1}$ (Wolframalpha result)

b. Considering two particles p_1 and p_2, the relative velocity of p_2 in the frame of p_1 (or the opposite) is given by :

$$v = \frac{v_{p_1} + v_{p_2}}{1 + \frac{v_{p_1} v_{p_2}}{c^2}}$$

Here we have : $v_{p_1} = v_{p_2} = v_{proton}$. If we do the computation we obtain $v = 2.9979245799999999 \times 10^8 \approx c$ (Wolframalpha result)

2.3

a. From the equipartition theorem we know that

$$< E >= \frac{1}{2} k_b T$$

withk_b the Boltzmann constant. Because the photon can have two polarization state we have

$$< E_{photon} >= k_b T = 2.353 \times 10^{-4} eV$$

. Wolframalpha result

b. We consider the following collision :

$$p + \gamma \longrightarrow p + \pi^0$$

We can write in the lab frame

$$\mathbf{p}_{T,initial} = \mathbf{p}_{p,min} + \mathbf{p}_{\gamma}$$

For the final state we have :

$$\mathbf{p}_{T,final} = \mathbf{p}_p + \mathbf{p}_{\pi^0}$$
$$\mathbf{p}_{T,final} = \begin{pmatrix} E_p/c \\ \vec{p}_p \end{pmatrix} + \begin{pmatrix} E_{\pi^0}/c \\ \vec{p}_{\pi^0} \end{pmatrix}$$

The minimum energy is required when particle are produce at rest in the center of mass frame, i.e. $\vec{p}_p + \vec{p}_{\pi^0} = \vec{0}$

$$\mathbf{p}_{T,final} = \begin{pmatrix} (E_p + E_{\pi^0})/c \\ \vec{0} \end{pmatrix} = \begin{pmatrix} (M_p + M_{\pi^0})c \\ \vec{0} \end{pmatrix}$$

The square o total four momentum is Lorentz invariant, i.e.

$$(\mathbf{p}^{LAB}_{T,initial})^2 = (\mathbf{p}^{COM}_{T,initial})^2 = (\mathbf{p}^{LAB}_{T,final})^2 = (\mathbf{p}^{COM}_{T,final})^2$$

So we have for the initial state :

$$(p_{T,initial})^2 = \mathbf{p}^2_{p,min} + 2\mathbf{p}_{p,min}\mathbf{p}_{\gamma} + \mathbf{p}^2_{\gamma}$$

From the definition of the four-momentum we have $\mathbf{p}^2_{p,min} = M_p^2 c^2$ and $\mathbf{p}^2_{\gamma} = 0$. We need to compute $2\mathbf{p}_{p,min}\mathbf{p}_{\gamma}$. In high relativistic regime we have $E^2_{p,min} = m_p^2 c^4 + \vec{p}^2_{p,min} c^2 \approx p^2_{p,min} c^2$ so $\vec{p}_{p,min} = E_{p,min}/c$. We therefor have for the proton four-momentum vector :

$$\mathbf{p}_{p,min} = \begin{pmatrix} E_{p,min}/c \\ E_{p,min}/c \end{pmatrix}$$

For the photon we obtain a similar vector with a minus sign for the momentum (as they collide) :

$$\mathbf{p}_{\gamma} = \begin{pmatrix} E_{\gamma}/c \\ -E_{\gamma}/c \end{pmatrix}$$

It leads to :

$$\begin{aligned} \mathbf{p}^2_{T,\text{initial}} &= \mathbf{p}^2_{p,\text{min}} + 2\mathbf{p}_{p,\text{min}} \cdot \mathbf{p}_{\gamma} \\ &= M_p^2 c^2 + \frac{1}{c^2}\left((E_{p,\text{min}} + E_{\gamma})^2 - (E_{p,\text{min}} - E_{\gamma})^2\right) \\ &= M_p^2 c^2 + \frac{4E_{p,\text{min}}E_{\gamma}}{c^2} \end{aligned}$$

For the final state we have :

$$(\mathbf{p}_{T,final})^2 = (M_p + M_{\pi^0})^2 c^2$$

So we obtain

$$\begin{aligned} M_p^2 c^2 + \frac{4E_{p,min}E_{\gamma}}{c^2} &= (M_p + M_{\pi^0})^2 c^2 \\ E_{p,min} &= \frac{((M_p + M_{\pi^0})^2 - M_p^2)c^4}{4E_{\gamma}} \\ E_{p,min} &\approx 2.885 \times 10^{20} eV \end{aligned}$$

(Wolframalpha result)

c. The relativistic momentum is expressed as : $p = \gamma m v$. Using the definition of gamma we can express the velocity as :

$$v = \frac{p}{\sqrt{m^2 + \frac{p^2}{c^2}}}$$

In the center of mass frame the outgoing particles are both produced at rest, which means that in the lab frame they have the same velocity, i.e :

$$\begin{aligned} v_p &= v_{\pi^0} \\ \frac{p_p}{\sqrt{M_p^2 + \frac{p_p^2}{c^2}}} &= \frac{p_{\pi^0}}{\sqrt{M_{\pi^0}^2 + \frac{p_{\pi^0}^2}{c^2}}} \\ p_p &= p_{\pi^0} \frac{M_p}{M_{\pi^0}} \end{aligned}$$

Momentum conservation gives us :

$$\begin{aligned} p_{p,initial} + p_{\gamma,initial} &= p_{p,final} + p_{\pi^0,final} \\ E_{p,min}/c - E_\gamma/c &= p_{p,final}(1 + \frac{M_{\pi^0}}{M_p}) \\ p_{p,final} &= \frac{E_{p,min}/c - E_\gamma/c}{(1 + \frac{M_{\pi^0}}{M_p})} \end{aligned}$$

The energy of the proton can be calculated with :

$$\begin{aligned} E_{p,final} &= \sqrt{M_p^2 c^4 + p_{p,final}^2 c^2} \\ &\approx 2.5221710^{20} eV \end{aligned}$$

(Wolframalpha result)

2.4

We consider the transformation :

$$Y : (t, x, y, z) \to (t, x, -y, z)$$

It is consider as a Lorentz transformation if it preserve the Minkowski metric, i.e. :

$$\Lambda^T g \Lambda = g$$

In our case $Y = diag(1, 1, -1, 1)$ clearly satisfy the relation. Moreover one can remark that :

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\pi & 0 & \sin\pi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\pi & 0 & \cos\pi \end{pmatrix}}_{R_y(\theta=\pi)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{P}$$

As a product of a discrete and continuous Lorentz transformation it is continuous one (i.e. not discrete).

2.5

The Compton scattering is the quantum theory of high frequency photons scattering following an interaction with a charged particle, usually an electron.

a. Specifically, when the photon hits electrons, it releases loosely bound electrons from the outer valence shells of atoms or molecules. In fact other interactions with the electron are negligible compared with the energy transferred between the photon and the electron (for X-rays the energies exchanged are in the keV range and vastly greater than typical electron binding energies in atoms).

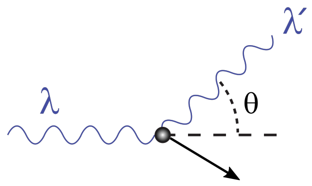

Figure 1: Compton Scattering

b. The momentum conservation gives us :

$$\begin{aligned} \vec{p}_{ei} + \vec{p}_{\gamma} &= \vec{p}_{ef} + \vec{p}_{\gamma'} \\ \vec{p}_{\gamma} &= \vec{p}_{ef} + \vec{p}_{\gamma'} \\ (\vec{p}_{\gamma} - \vec{p}_{\gamma'})^2 &= \vec{p}_{ef}^{\,2} \\ p_{\gamma}^2 - 2(\vec{p}_{\gamma} \cdot \vec{p}_{\gamma'}) + p_{\gamma'}^2 &= p_{ef}^2 \\ p_{\gamma}^2 - 2p_{\gamma}p_{\gamma'}\cos\theta + p_{\gamma'}^2 &= p_{ef}^2 \end{aligned}$$

From energy conservation we also have :

$$\begin{aligned} E_{ei} + E_{\gamma} &= E_{ef} + E_{\gamma'} \\ m_e c^2 + p_{\gamma} c &= \sqrt{m_e^2 c^4 + p_{ef}^2 c^2} + p_{\gamma'} c \\ (m_e c^2 + p_{\gamma} c - p_{\gamma'} c)^2 - m_e^2 c^4 &= p_{ef}^2 c^2 \\ p_{\gamma}^2 + p_{\gamma'}^2 + 2m_e c p_{\gamma} - 2m_e c p_{\gamma'} - 2p_{\gamma}p_{\gamma'} &= p_{ef}^2 \end{aligned}$$

Replacing p_{ef}^2 we obtain:

$$\begin{aligned} -2p_{\gamma}p_{\gamma'}\cos\theta &= +2m_e c p_{\gamma} - 2m_e c p_{\gamma'} - 2p_{\gamma}p_{\gamma'} \\ 2p_{\gamma}p_{\gamma'}(1-\cos\theta) &= 2m_e c(p_{\gamma} - p_{\gamma'}) \\ \frac{(1-\cos\theta)}{m_e c} &= \left(\frac{1}{p_{\gamma'}} - \frac{1}{p_{\gamma}}\right) \end{aligned}$$

From the momentum relation $p = hf/c = h/\lambda$ we obtain:

$$\frac{h(1-\cos\theta)}{m_e c} = \lambda' - \lambda = \Delta\lambda$$

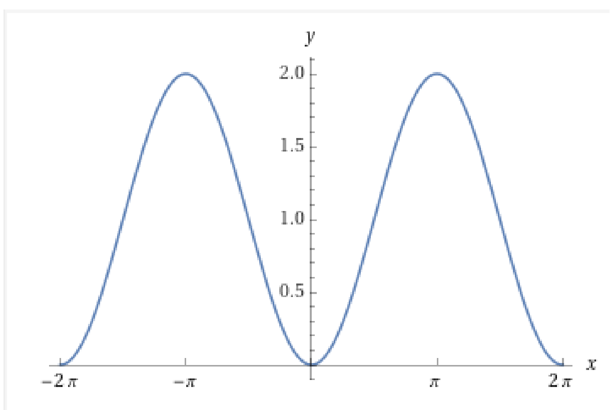

Figure 2: Wavelenght dependency $(1-\cos\theta)$

c.

$$2p_\gamma p_{\gamma'}(1-\cos\theta) = 0$$

So $\theta = 0$, and therefor $\Delta\lambda = 0$. If the electron mass was zero then the wavelength, and so the energy of the reflected X-ray wouldn't change.

d. If we didn't considered the quantized photon momenta, i.e. using classical EM theory, the scattering would be describe by the Thomson scattering. The incident wave accelerates the charged particle, which will emit radiation at the same frequency as the incident wave, and thus the wave is scattered. The particle's kinetic energy and photon frequency do not change as a result of the scattering, so we would expect a uniform distribution.

2.6

a. The four momentum vector is given by :

$$\mathbf{k} = \begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} dk^0 \delta(\mathbf{k}^2 - m^2)\theta(k^0) &= \int_{-\infty}^{+\infty} dk^0 \delta({k^0}^2 - \vec{k}^2 - m^2)\theta(k^0) \\
&= \int_0^{+\infty} dk^0 \delta({k^0}^2 - \vec{k}^2 - m^2)
\end{aligned}$$

Using the properties :

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|},$$

with $x_i = \pm\sqrt{\vec{k}^2 + m^2}$ We obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty} dk^0 \delta(\mathbf{k}^2 - m^2)\theta(k^0) &= \int_0^{+\infty} dk^0 \frac{\delta(k^0 + \sqrt{\vec{k}^2 + m^2}) + \delta(k^0 - \sqrt{\vec{k}^2 + m^2})}{|2\sqrt{\vec{k}^2 + m^2}|} \\
&= \frac{1}{2\sqrt{\vec{k}^2 + m^2}} \int_0^{+\infty} dk^0 \delta(k^0 - \sqrt{\vec{k}^2 + m^2}) \\
&= \frac{1}{2\sqrt{\vec{k}^2 + m^2}}
\end{aligned}$$

b. The four vector transform as :

$$k \to k' = \Lambda k$$

with Λ the Lorentz transformation. The 4-volume element transforms as:

$$d^4k \to d^4k' = J d^4k$$

with J the Jacobian determinant of the coordinate transformation. Here we have

$$d^4k' = |\det(\Lambda)|\, d^4k = d^4k$$

So d^4k is Lorentz invariant.

c. From (a.) we know that :

$$\begin{aligned}
\int \frac{d^3k}{2w_k} &= \int d^3k \int dk^0 \delta(\mathbf{k}^2 - m^2)\theta(k^0) \\
&= \int d^4k\ \delta(\mathbf{k}^2 - m^2)\theta(k^0)
\end{aligned}$$

From (b.) we know that d^4k is Lorentz invariant. Moreover $\delta(\mathbf{k}^2 - m^2)\theta(k^0)$ is also Lorentz invariant as a scalar function. So the integral is Lorentz invariant.

2.7

a.

$$\begin{aligned}
\partial z(e^{-za^\dagger} a e^{za^\dagger}) &= -a^\dagger e^{-za^\dagger} a e^{za^\dagger} + e^{-za^\dagger} a a^\dagger e^{z\ a^\dagger} \\
&= -e^{-za^\dagger} a^\dagger a e^{za^\dagger} + e^{-za^\dagger} a a^\dagger e^{z\ a^\dagger} \text{(because } a^\dagger \text{ commute with himself...)} \\
&= e^{-za^\dagger}(aa^\dagger - a^\dagger a) e^{za^\dagger} \\
&= e^{-za^\dagger} e^{za^\dagger} = e^{-za^\dagger + za^\dagger} \text{(same argument)} \\
&= 1
\end{aligned}$$

b. From (a.) we know that :

$$e^{-za^\dagger} a e^{za^\dagger} = z + C$$

For $z = 0$, we get $C = a$.

$$\begin{aligned}
e^{-za^\dagger} a e^{za^\dagger} &= z + a \\
a e^{za^\dagger} &= z e^{za^\dagger} + e^{za^\dagger} a
\end{aligned}$$

So we have :

$$\begin{aligned}
a\,|z\rangle &= a e^{za^\dagger}\,|0\rangle \\
&= (z e^{za^\dagger} + e^{za^\dagger} a)\,|0\rangle \\
&= z e^{za^\dagger}\,|0\rangle + e^{za^\dagger} a)\,|0\rangle \\
&= z e^{za^\dagger}\,|0\rangle = z\,|z\rangle
\end{aligned}$$

c.

$$\langle n| = \langle 0| \frac{a^n}{\sqrt{n!}}$$

So :

$$\begin{aligned}
\langle n|z\rangle &= \langle 0| \frac{a^n}{\sqrt{n!}}\,|z\rangle \\
&= \frac{1}{\sqrt{n!}} \langle 0|\, a^n\,|z\rangle \\
&= \frac{1}{\sqrt{n!}} \langle 0|\, z^n\,|z\rangle \\
&= \frac{z^n}{\sqrt{n!}} \langle 0|\, e^{za^\dagger}\,|0\rangle \\
&= \frac{z^n}{\sqrt{n!}} \langle 0|\, (\sum \frac{(za^\dagger)^n}{n!})\,|0\rangle \\
&= \frac{z^n}{\sqrt{n!}} \langle 0|\, (1 + za^\dagger + \frac{(za^\dagger)^2}{2!} + ...)\,|0\rangle \\
&= \frac{z^n}{\sqrt{n!}}
\end{aligned}$$

d We have :

$$\begin{aligned}
\langle q\rangle &= \frac{\langle z|\, q\,|z\rangle}{\langle z|z\rangle} = \frac{1}{\sqrt{2}} \frac{\langle z|\, a + a^\dagger\,|z\rangle}{\langle z|z\rangle} \\
\langle p\rangle &= \frac{\langle z|\, p\,|z\rangle}{\langle z|z\rangle} = \frac{1}{\imath\sqrt{2}} \frac{\langle z|\, a - a^\dagger\,|z\rangle}{\langle z|z\rangle}
\end{aligned}$$

First lets compute $\langle z|z\rangle$

$$\begin{aligned}\langle z|z\rangle &= \langle 0|\, e^{\bar{z}a}\,|z\rangle \\ &= \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}\,|z\rangle \\ &= \langle 0|\sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}a^k\,|z\rangle \\ &= \langle 0|\sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}z^k\,|z\rangle \\ &= \sum_{k=0}^{+\infty}\frac{|z|^{2k}}{k!}\,\langle 0|z\rangle \\ &= e^{|z|^2}\end{aligned}$$

Now we compute $\langle z|\,q\,|z\rangle$:

$$\langle z|\,q\,|z\rangle = \frac{1}{\sqrt{2}}\,\langle z|\,a+a^\dagger\,|z\rangle = \frac{1}{\sqrt{2}}(\langle z|\,a\,|z\rangle + \langle z|\,a^\dagger\,|z\rangle)$$

$$\langle z|\,a\,|z\rangle = \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}a\,|z\rangle = z\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}\,\langle 0|z\rangle = ze^{|z|^2}$$

$$\langle z|\,a^2\,|z\rangle = \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}a^2\,|z\rangle = z^2\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}\,\langle 0|z\rangle = z^2e^{|z|^2}$$

$$\begin{aligned}\langle z|\,a^\dagger\,|z\rangle = \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}a^\dagger\,|z\rangle &= \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle 0|\,a^k a^\dagger\,|z\rangle = \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle k-1|\,\sqrt{k}\sqrt{k!}\,|z\rangle \\ &= \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\frac{k\sqrt{(k-1)!}z^{k-1}}{\sqrt{(k-1)!}} = \bar{z}\sum_{k=1}^{+\infty}\frac{(\bar{z}z)^{k-1}}{(k-1)!} = \bar{z}e^{|z|^2}\end{aligned}$$

$$\begin{aligned}\langle z|\,{a^\dagger}^2\,|z\rangle = \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}{a^\dagger}^2\,|z\rangle &= \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle 0|\,a^k{a^\dagger}^2\,|z\rangle = \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle k-2|\,\sqrt{k(k-1)}\sqrt{k!}\,|z\rangle \\ &= \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\frac{k(k-1)\sqrt{(k-2)!}z^{k-2}}{\sqrt{(k-2)!}} = \bar{z}^2\sum_{k=2}^{+\infty}\frac{(\bar{z}z)^{k-2}}{(k-2)!} = \bar{z}^2e^{|z|^2}\end{aligned}$$

$$\langle z|\,q^2\,|z\rangle = \frac{1}{\sqrt{2}}\,\langle z|\,(a+a^\dagger)^2\,|z\rangle = \frac{1}{\sqrt{2}}(\langle z|\,a^2\,|z\rangle + \langle z|\,aa^\dagger\,|z\rangle + \langle z|\,a^\dagger a\,|z\rangle + \langle z|\,(a^\dagger)^2\,|z\rangle)$$

$$\begin{aligned}\langle z|\,aa^\dagger\,|z\rangle &= \langle 0|\sum_{k=0}^{+\infty}\frac{(\bar{z}a)^k}{k!}aa^\dagger\,|z\rangle = \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle 0|\,a^{k+1}a^\dagger\,|z\rangle = \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\,\langle k|\,\sqrt{k+1}\sqrt{(k+1)!}\,|z\rangle \\ &= \sum_{k=0}^{+\infty}\frac{\bar{z}^k}{k!}\frac{(k+1)\sqrt{k!}z^k}{\sqrt{k!}} = \bar{z}z\sum_{k=1}^{+\infty}\frac{(\bar{z}z)^{k-1}}{(k-1)!} + \sum_{k=0}^{+\infty}\frac{(\bar{z}z)^k}{k!} = (|z|^2+1)e^{|z|^2}\end{aligned}$$

$$\langle z| a^\dagger a |z\rangle = \langle 0| \sum_{k=0}^{+\infty} \frac{(\bar{z}a)^k}{k!} a^\dagger a |z\rangle = \sum_{k=0}^{+\infty} \frac{\bar{z}^k}{k!} z \langle 0| a^k a^\dagger |z\rangle = \sum_{k=0}^{+\infty} \frac{\bar{z}^k}{k!} z \langle k-1| \sqrt{k}\sqrt{k!} |z\rangle$$
$$= \sum_{k=0}^{+\infty} \frac{\bar{z}^k z^k}{k!} \frac{k\sqrt{(k-1)!}}{\sqrt{(k-1)!}} = \bar{z}z \sum_{k=1}^{+\infty} \frac{(\bar{z}z)^{k-1}}{(k-1)!} = |z|^2 e^{|z|^2}$$

Finally we obtain for :

$$\Delta q^2 = \frac{\langle q^2\rangle - \langle q\rangle^2}{2} = \frac{z^2 + \bar{z}^2 + 2|z|^2 + 1 - (z+\bar{z})^2}{2} = \frac{1}{2}$$
$$\Delta p^2 = \frac{\langle p^2\rangle - \langle p\rangle^2}{2} = \frac{z^2 + \bar{z}^2 - 2|z|^2 + 1 - (z-\bar{z})^2}{2} = \frac{1}{2}$$

Therefor we have

$$\Delta p \Delta q = \frac{1}{2}$$

e. Lets suppose we have an eigenstate of $a^\dagger$ express as $|z'\rangle = \sum_n C_n |n\rangle$ with an eigenvalue λ

$$a^\dagger |z'\rangle = a^\dagger \sum_{n\geq 0} C_n |n\rangle$$
$$\lambda |z'\rangle = \sum_{n\geq 0} C_n \sqrt{n+1} |n+1\rangle$$
$$\lambda \sum_n C_n |n\rangle = \sum_{n\geq 0} C_n \sqrt{n+1} |n+1\rangle$$
$$\lambda \sum_n C_n |n\rangle = \sum_{n\geq 1} C_{n-1} \sqrt{n} |n\rangle$$
$$C_n = \frac{C_{n-1}\sqrt{n}}{\lambda} = \frac{C_0\sqrt{n!}}{\lambda^n}$$

For $\lambda \neq 0$ we have

$$C_0 = \langle 0|z'\rangle = \frac{1}{\lambda} \langle 0| a^\dagger |z'\rangle$$
$$= \sum_{n\geq 1} C_{n-1}\sqrt{n} \langle 0|n\rangle$$
$$= 0$$

Therefor there is no eigenstates for $a^\dagger$.

3 Classical field theory